

Q-FANO THREEFOLDS WITH THREE BIRATIONAL MORI FIBER STRUCTURES

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ABSTRACT. In this paper we give first examples of \mathbb{Q} -Fano threefolds whose birational Mori fiber structures consist of exactly three \mathbb{Q} -Fano threefolds. These examples are constructed as weighted hypersurfaces in a specific weighted projective space. We also observe that the number of birational Mori fiber structures does not behave upper semi-continuously in a family of \mathbb{Q} -Fano threefolds.

1. INTRODUCTION

A Mori fiber space which is birational to a given variety is called a *birational Mori fiber structure* of the variety. We say that a \mathbb{Q} -Fano variety X with Picard number one is *birationally rigid* (resp. *birationally birigid*) if the birational Mori fiber structures of X consist of one element X (resp. exactly two elements including X). There are many birationally rigid \mathbb{Q} -Fano varieties such as nonsingular hypersurfaces of degree $n + 1$ in \mathbb{P}^{n+1} for $n \geq 3$ ([12, 8]) and quasismooth anticanonically embedded \mathbb{Q} -Fano threefold weighted hypersurfaces ([7, 5]). Compared to birational rigidity, \mathbb{Q} -Fano varieties with finite birational Mori fiber structures (or with finite pliability) are less known. Corti-Mella [6] proved that a quartic threefold with a specific singular point is birationally birigid. Cheltsov-Grinenko [5] constructed an example of a birationally birigid complete intersection of a quadric and a cubic in \mathbb{P}^5 with a single ordinary double point. In [15, 16], we proved that 14 families of \mathbb{Q} -Fano threefold weighted complete intersections consist of birationally birigid \mathbb{Q} -Fano threefolds. There are other interesting examples of birationally non-rigid \mathbb{Q} -Fano threefolds [1, 2, 3] but their birational Mori fiber structures are yet to be determined.

The aim of this paper is to construct first examples of \mathbb{Q} -Fano threefolds with exactly three birational Mori fiber structures. We also observe that the number of birational Mori fiber structures does not behave upper semi-continuously in a family. The main objects of this paper are weighted hypersurfaces of degree 8 in the weighted projective space $\mathbb{P}(1, 1, 2, 2, 3)$. We explain known results for this family.

Theorem 1.1 ([5, 7]). *A quasismooth weighted hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$ is birationally rigid.*

Theorem 1.2 ([16]). *A \mathbb{Q} -Fano weighted hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$ with a single $cAx/2$ singular point together with some other terminal quotient singular points is birationally birigid. More precisely, it is birational to a quasismooth \mathbb{Q} -Fano weighted complete intersection of type $(6, 8)$ in $\mathbb{P}(1, 1, 2, 3, 4, 4)$ and it is not birational to any other Mori fiber space.*

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We consider further special members that admit two $cAx/2$ singular points and determine the birational Mori fiber structures of them. We state the main theorem of this paper. In the statement, $\mathbb{P}(1, 1, 2, 2, 3)$ (resp. $\mathbb{P}(1, 1, 2, 3, 4, 4)$) is the weighted projective space with homogeneous coordinates x_0, x_1, y_0, y_1 and z of degree respectively 1, 1, 2, 2 and 3 (resp. x_0, x_1, y, z, s_0 and s_1 of degree respectively 1, 1, 2, 3, 4 and 4).

Theorem 1.3. *Let X' be a \mathbb{Q} -Fano weighted hypersurface*

$$X' = (y_0^2 y_1^2 + y_0 a_6 + y_1 b_6 + c_8 = 0) \subset \mathbb{P}(1, 1, 2, 2, 3),$$

where $a_6, b_6, c_8 \in \mathbb{C}[x_0, x_1, z]$ are homogeneous polynomials of degree respectively 6, 6, 8. Then X' is birational to \mathbb{Q} -Fano weighted complete intersections

$$X_1 = (s_0 y + s_1 y + a_6 = s_0 s_1 - y b_6 - c_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4),$$

and

$$X_2 = (s_0 y + s_1 y + b_6 = s_0 s_1 - y a_6 - c_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4),$$

and not birational to any other Mori fiber space. Moreover we have the following.

- (1) If (a_6, b_6, c_8) is asymmetric (see Definition 2.12), then X_1 is not isomorphic to X_2 and the birational Mori fiber structures of X' consist of three \mathbb{Q} -Fano threefolds X', X_1 and X_2 .
- (2) If (a_6, b_6, c_8) is symmetric, then X_1 is isomorphic to X_2 and the birational Mori fiber structures of X' consist of two \mathbb{Q} -Fano threefolds X' and $X_1 \cong X_2$.

In the above theorem, the members X' with the property (1) are more general than those with the property (2). We observe through the above theorems that the number of birational Mori fiber structures increases as we specialize \mathbb{Q} -Fano threefolds in a family except for the case (2) in Theorem 1.3 where the number decreases. Therefore the number of birational Mori fiber structures does not behave upper semi-continuously in a family. A similar observation is also given in [4].

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2. PRELIMINARIES

2.1. Quasismoothness. Let $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space with homogeneous coordinates x_0, \dots, x_n . We assume that \mathbb{P} is well-formed, that is, $\gcd(a_0, \dots, \hat{a}_i, \dots, a_n) = 1$ for each i , and let X be a closed subvariety of \mathbb{P} . For a non-empty subset $I = \{i_1, \dots, i_k\}$ of $\{0, \dots, n\}$, we define

$$\Pi_I^\circ = \left(\bigcap_{i \in I} (x_i \neq 0) \right) \cap \left(\bigcap_{j \notin I} (x_j = 0) \right) \subset \mathbb{P}$$

and call it a *coordinate stratum* of \mathbb{P} with respect to I . For a $(k+1)$ -tuple of non-negative integers $m = (m_0, \dots, m_k)$, we write

$$x_I^m = x_{i_0}^{m_0} \cdots x_{i_k}^{m_k}.$$

Definition 2.1. Let X be a closed subscheme of \mathbb{P} and $p: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}$ the natural projection. We say that X is *quasismooth* if the affine cone $C_X \subset \mathbb{A}^{n+1}$ of X , which is the closure of $p^{-1}(X)$ in \mathbb{A}^{n+1} , is smooth outside the origin. For a non-empty subset $I \subset \{0, \dots, n\}$, we say that X is *quasismooth along Π_I°* if C_X is smooth along $p^{-1}(\Pi_I^\circ)$.

It follows from the definition that a closed subscheme $X \subset \mathbb{P}$ is quasismooth if and only if X is quasismooth along Π_I° for any non-empty subset of $I \subset \{0, \dots, n\}$.

Definition 2.2. Let M be a set of monomials of degree d . We denote by $\Lambda(M)$ the linear system on \mathbb{P} spanned by elements in M . Let M_1 and M_2 be sets of monomials of degree respectively d_1 and d_2 . We define

$$\Lambda(M_1, M_2) = \{X_1 \cap X_2 \subset \mathbb{P} \mid X_1 \in \Lambda(M_1), X_2 \in \Lambda(M_2)\},$$

which is the family of weighted complete intersections of type (d_1, d_2) defined as the scheme-theoretic intersection of weighted hypersurfaces in $\Lambda(M_1)$ and $\Lambda(M_2)$.

We re-state the results of [11] on quasismoothness of weighted complete intersections in a generalized form. Although the statements are slightly different from the original ones, proofs are completely parallel. More precisely, the proofs can be done by replacing complete linear systems of degree d, d_1, d_2 with linear systems $\Lambda(M), \Lambda(M_1), \Lambda(M_2)$, respectively, in the proofs of the corresponding theorems in [11]. A weighted hypersurface of degree d is said to be a *linear cone* if its defining polynomial f can be written as $f = \alpha x_i + (\text{other terms})$ for some i and non-zero $\alpha \in \mathbb{C}$.

Theorem 2.3 (cf. [11, 8.1 Theorem]). *Let $I = \{i_0, \dots, i_{k-1}\}$ be a non-empty subset of $\{0, \dots, n\}$ and M a set of monomials of degree d . A general weighted hypersurface in $\Lambda(M)$ which is not a linear cone is quasismooth along Π_I° if one of the following assertions hold.*

- (1) *There exists a monomial $x_I^m = x_{i_0}^{m_0} \dots x_{i_{k-1}}^{m_{k-1}} \in M$.*
- (2) *For $\mu = 1, \dots, k$, there exist monomials*

$$x_I^{m_\mu} x_{e_\mu} = x_{i_0}^{m_{0,\mu}} \dots x_{i_{k-1}}^{m_{k-1,\mu}} x_{e_\mu} \in M,$$

where $\{e_\mu\}$ are k distinct elements.

Theorem 2.4 (cf. [11, 8.7 Theorem]). *Let $I = \{i_0, \dots, i_{k-1}\}$ be a non-empty subset of $\{0, \dots, n\}$ and M_1, M_2 sets of monomials of degree d_1, d_2 , respectively. A general weighted complete intersection in $\Lambda(M_1, M_2)$ which is not the intersection of a linear cone with another hypersurface is quasismooth along Π_I° if one of the following assertions hold.*

- (1) *There exist monomials $x_I^{m_1} \in M_1$ and $x_I^{m_2} \in M_2$.*
- (2) *There exists a monomial $x_I^m \in M_1$, and for $\mu = 1, \dots, k-1$ there exist monomials $x_I^{m_\mu} x_{e_\mu} \in M_2$, where $\{e_\mu\}$ are $k-1$ distinct elements.*
- (3) *There exists a monomial $x_I^m \in M_2$, and for $\mu = 1, \dots, k-1$ there exist monomials $x_I^{m_\mu} x_{e_\mu} \in M_1$, where $\{e_\mu\}$ are $k-1$ distinct elements.*
- (4) *For $\mu = 1, \dots, k$, there exist monomials $x_I^{m_\mu^1} x_{e_\mu^1} \in M_1$, and $x_I^{m_\mu^2} x_{e_\mu^2} \in M_2$, such that $\{e_\mu^1\}$ are k -distinct elements, $\{e_\mu^2\}$ are k distinct elements and $\{e_\mu^1, e_\mu^2\}$ contains at least $k+1$ distinct elements.*

Let $\mathbb{P} := \mathbb{P}(a_0, \dots, a_4)$ be a weighted projective space with homogeneous coordinates x_0, x_1, x_2, x_3, x_4 with $\deg x_i = a_i$ and V a weighted hypersurface in \mathbb{P} which contains a weighted complete intersection curve $(x_0 = f = g = 0)$, where $f, g \in \mathbb{C}[x_1, x_2, x_3, x_4]$ with $\deg f \leq \deg g =: m$. We give a criterion of quasismoothness of a general member of a linear system on V along Γ . We set $R = \mathbb{C}[x_0, \dots, x_4]$ and denote by R_n the degree n part of the homogeneous ring R for a non-negative integer n . Let $\mathcal{M} \subset |\mathcal{O}_V(m)|$ be a linear system on V generated by homogeneous polynomials h_1, \dots, h_l of degree m and $R_{\mathcal{M}} \subset R_m$ the subspace spanned by h_1, \dots, h_l . For a homogeneous polynomial $h \in R$ of degree $d \leq m$, we define \mathcal{M}_h to be the linear system on V generated by $\Phi_h^{-1}(R_{\mathcal{M}})$, where $\Phi_h: R_{m-d} \rightarrow R_m$ is the map defined by multiplying h . We denote by $\text{NQsm}(V)$ the non-quasismooth locus of V .

Lemma 2.5. *Let $V \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_4)$ and $\Gamma = (x_0 = f = g = 0) \subset V$, $\deg f \leq \deg g =: m$, be as above. Suppose that Γ is quasismooth and that $\text{Bs } \mathcal{M}_f \not\supset \Gamma$. Then a general member of \mathcal{M} is quasismooth along $\Gamma \setminus (\text{NQsm}(V) \cup \text{Bs } \mathcal{M}_{x_0})$.*

Proof. The defining polynomial of V can be written as $bf + cg + x_0h$ for some $b, c \in \mathbb{C}[x_1, \dots, x_4]$ and $h \in \mathbb{C}[x_0, \dots, x_4]$. Let $S \in \mathcal{M}$ be a general member. The section s which cuts out S on V can be written as $s = df + \alpha g + x_0e$ for some $\alpha \in \mathbb{C}$, $d \in \mathbb{C}[x_1, \dots, x_4]$ and $e \in \mathbb{C}[x_0, \dots, x_4]$. The restriction of the Jacobian matrix of the affine cone C_S of S can be computed as

$$J_{C_S}|_{\Gamma} = \begin{pmatrix} h & b \frac{\partial f}{\partial x_1} + c \frac{\partial g}{\partial x_1} & b \frac{\partial f}{\partial x_2} + c \frac{\partial g}{\partial x_2} & b \frac{\partial f}{\partial x_3} + c \frac{\partial g}{\partial x_3} & b \frac{\partial f}{\partial x_4} + c \frac{\partial g}{\partial x_4} \\ e & d \frac{\partial f}{\partial x_1} + \alpha \frac{\partial g}{\partial x_1} & d \frac{\partial f}{\partial x_2} + \alpha \frac{\partial g}{\partial x_2} & d \frac{\partial f}{\partial x_3} + \alpha \frac{\partial g}{\partial x_3} & d \frac{\partial f}{\partial x_4} + \alpha \frac{\partial g}{\partial x_4} \end{pmatrix}.$$

Note that the matrix

$$\begin{pmatrix} b \frac{\partial f}{\partial x_1} + c \frac{\partial g}{\partial x_1} & b \frac{\partial f}{\partial x_2} + c \frac{\partial g}{\partial x_2} & b \frac{\partial f}{\partial x_3} + c \frac{\partial g}{\partial x_3} & b \frac{\partial f}{\partial x_4} + c \frac{\partial g}{\partial x_4} \\ d \frac{\partial f}{\partial x_1} + \alpha \frac{\partial g}{\partial x_1} & d \frac{\partial f}{\partial x_2} + \alpha \frac{\partial g}{\partial x_2} & d \frac{\partial f}{\partial x_3} + \alpha \frac{\partial g}{\partial x_3} & d \frac{\partial f}{\partial x_4} + \alpha \frac{\partial g}{\partial x_4} \end{pmatrix} \\ = \begin{pmatrix} b & c \\ d & \alpha \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & \frac{\partial g}{\partial x_4} \end{pmatrix}.$$

is of rank 2 at any point of $\Gamma \setminus (\alpha b - cd = 0)$ and is of rank 1 at any point of $\Gamma \cap (\alpha b - cd = 0)$ since Γ is quasismooth. It follows that S is quasismooth along $\Gamma \setminus (\alpha b - cd = 0)$. We shall show that $J_{C_S}|_{\Gamma}$ is of rank 2 at any point $\mathbf{p} \in \Gamma \setminus (\text{NQsm}(V) \cup \text{Bs } \mathcal{M}_{x_0})$.

Assume that $(b = c = 0) \cap \Gamma = \Gamma$, that is, both b and c vanish along Γ . Then h does not vanish at \mathbf{p} since V is quasismooth at \mathbf{p} . It follows that $J_{C_S}|_{\Gamma}$ is of rank 2 at \mathbf{p} .

In the following, we assume that $(b = c = 0) \cap \Gamma \neq \Gamma$. We claim that $(\alpha b - cd = 0) \cap \Gamma$ is a finite set of points. If $(b = 0) \not\supset \Gamma$, then $(\alpha b - cd = 0) \cap \Gamma \neq \Gamma$ for a general choice of α and d . Assume that $(b = 0) \supset \Gamma$. Then $(c = 0) \not\supset \Gamma$ since $(b = c = 0) \cap \Gamma \neq \Gamma$. In this case $(\alpha b - cd = 0) \cap \Gamma = (cd = 0) \cap \Gamma$ and it is a finite set of points since $d \in \mathcal{M}_f$ and $\text{Bs } \mathcal{M}_f \not\supset \Gamma$ by the assumption.

If $\mathbf{p} \notin (\alpha b - cd = 0)$, then $J_{C_S}|_{\Gamma}$ is of rank 2 at \mathbf{p} by the above argument. It remains to consider the case $\mathbf{p} \in (\alpha b - cd = 0) \cap \Gamma$. We see that $J_{C_S}|_{\Gamma}$ is of rank 2 at \mathbf{p} for a general choice of e since $e \in \mathcal{M}_{x_0}$ and $\mathbf{p} \notin \text{Bs } \mathcal{M}_{x_0}$. Since there are only finitely many points in $\Gamma \cap (\alpha b - cd = 0)$, this shows that $J_{C_S}|_{\Gamma}$ is of rank 2 at every point of $\Gamma \setminus (\text{NQsm}(V) \cup \text{Bs } \mathcal{M}_{x_0})$. This completes the proof. \square

2.2. Generality conditions. In the rest of this paper the coordinates $x_0, x_1, y_0, y_1, y, z, s_0$ and s_1 are of degree respectively 1, 1, 2, 2, 2, 3, 4 and 4. We set

$$\mathbb{P}(1, 1, 2, 2, 3) = \text{Proj } \mathbb{C}[x_0, x_1, y_0, y_1, z]$$

and

$$\mathbb{P}(1, 1, 2, 3, 4, 4) = \text{Proj } \mathbb{C}[x_0, x_1, y, z, s_0, s_1].$$

Let a_6, b_6 and c_8 be homogeneous polynomials of degree 6, 6 and 8, respectively, in variables x_0, x_1, z . We define weighted hypersurface

$$X' = (y_0^2 y_1^2 + y_0 a_6 + y_1 b_6 + c_8 = 0) \subset \mathbb{P}(1, 1, 2, 2, 3)$$

and weighted complete intersections

$$X_1 = (s_0 y + s_1 y + a_6 = s_0 s_1 - y b_6 - c_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4),$$

$$X_2 = (s_0 y + s_1 y + b_6 = s_0 s_1 - y a_6 - c_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4).$$

We define points of X' as

$$\mathbf{p}'_1 = (0:0:1:0:0), \mathbf{p}'_2 = (0:0:0:1:0), \mathbf{p}'_3 = (0:0:0:0:1),$$

and points of $X_i, i = 1, 2$, as

$$\mathbf{p}_1 = (0:0:0:0:1:0), \mathbf{p}_2 = (0:0:0:0:0:1), \mathbf{p}_3 = (0:0:1:0:0:0).$$

We introduce the following conditions on the triplet (a_6, b_6, c_8) .

- Condition 2.6.** (1) X' is quasismooth outside the points \mathbf{p}'_1 and \mathbf{p}'_2 .
 (2) The singularities of X' at \mathbf{p}'_1 and \mathbf{p}'_2 are both of type $cAx/2$.
 (3) Both X_1 and X_2 are quasismooth outside the point \mathbf{p}_3 .
 (4) The singularities of X_1 and X_2 at \mathbf{p}_3 are both of type $cAx/2$.

We shall show that this condition is satisfied for a general (a_6, b_6, c_8) .

Lemma 2.7. Condition 2.6 is satisfied for a general triplet (a_6, b_6, c_8) .

Proof. For a positive integer d and a monomial g , we define

$$M_d = \{x_0^k x_1^l z^m \mid k, l, m \geq 0 \text{ and } k + l + 3m = d\} \text{ and } gM_d = \{gh \mid h \in M_d\}.$$

We set

$$N' = \{y_0^2 y_1^2\} \cup y_0 M_6 \cup y_1 M_6 \cup M_8,$$

$$N_6 = \{s_0 y, s_1 y\} \cup M_6,$$

$$N_8 = \{s_0 s_1\} \cup y M_6 \cup M_8.$$

To verify conditions (1) and (3), it is enough to show that general members of $\Lambda(N')$ and $\Lambda(N_6, N_8)$ are quasismooth outside $\mathbf{p}'_1, \mathbf{p}'_2$ and \mathbf{p}_3 , respectively. But this follows from Theorems 2.3 and 2.4.

Note that $(a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is quasismooth for a general a_6 . We claim that if $(a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is quasismooth, then $\mathbf{p}'_1 \in X'$ is of type $cAx/2$. We work on the open subset where $y_0 \neq 0$. Then, by setting $y_0 = 1$, X' is defined as

$$(y_1^2 + a_6 + y_1 b_6 + c_8 = 0) \subset \mathbb{A}_{x_0, x_1, y_1, z}^4 / \mathbb{Z}_2(1, 1, 0, 1).$$

Since $(a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is quasismooth, $z^2 \in a_6$, and hence we may write $a_6 = z^2 + f_6(x_0, x_1)$ for some f_6 after replacing z . It follows again from quasismoothness of $(a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ that f_6 does not have a multiple component. By the

suitable analytic coordinate change, the germ (X', \mathbf{p}'_1) is analytically equivalent to the origin of

$$(y_1^2 + z^2 + g(x_0, x_1) = 0) \subset \mathbb{A}^4/\mathbb{Z}_2(1, 1, 0, 1),$$

where the lowest weight term of g is f_6 . It is easy to see that the singularity is terminal since f_6 does not have a multiple component. This shows that \mathbf{p}'_1 is of type $cAx/2$. By the symmetric argument, the point $\mathbf{p}'_2 \in X'$ is of type $cAx/2$ if $(b_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is quasismooth, and the condition (2) is verified.

We claim that the singularity of X_2 at \mathbf{p}_3 is equivalent to that of X' at \mathbf{p}'_1 . By setting $y = 1$ in the defining polynomials of X_2 , we see that (X_2, \mathbf{p}_3) is isomorphic to

$$\begin{aligned} (s_0 + s_1 + b_6 = s_0 s_1 - a_6 - c_8 = 0) &\subset \mathbb{A}_{x_0, x_1, z, s_0, s_1}^5 / \mathbb{Z}_2(1, 1, 1, 0, 0) \\ &\cong (s_0^2 + a_6 + s_0 b_6 + c_8 = 0) \subset \mathbb{A}_{x_0, x_1, s_0, z}^4 / \mathbb{Z}_2(1, 1, 0, 1). \end{aligned}$$

Hence the germ (X_2, \mathbf{p}_3) is isomorphic to (X_1, \mathbf{p}'_1) . By symmetry, we have $(X_1, \mathbf{p}_3) \cong (X', \mathbf{p}'_2)$. Therefore the condition (4) follows from (2). This completes the proof. \square

In the following we assume that (a_6, b_6, c_8) satisfies Condition 2.6. We see that $\text{Sing}(X') = \{\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3\}$ and the singularity of X' at \mathbf{p}'_3 is of type $\frac{1}{3}(1, 1, 2)$, and $\text{Sing}(X_i) = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and the singularity of X_i at $\mathbf{p}_1, \mathbf{p}_2$ are of type $\frac{1}{4}(1, 1, 4)$.

Lemma 2.8. *The following assertions hold.*

- (1) *The weighted hypersurfaces*

$$(a_6 = 0) \subset \mathbb{P}(1, 1, 3) \text{ and } (b_6 = 0) \subset \mathbb{P}(1, 1, 3)$$

are quasismooth.

- (2) *Let X be one of X', X_1 and X_2 , and \mathbf{p} a singular point of X . Then there is a unique extremal divisorial extraction centered at \mathbf{p} .*

Proof. Assume that $C := (a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is not quasismooth at a point $(\xi_0 : \xi_1 : \zeta) \in C$. Let σ be a complex number such that $\sigma^2 = c_8(\xi_0, \xi_1, \zeta)$ and set $\mathbf{p} = (\xi_0 : \xi_1 : 0 : \zeta : \sigma : -\sigma)$. We see that $\mathbf{p} \in X_1$ and X_1 is not quasismooth at \mathbf{p} . This is a contradiction because X_1 is quasismooth except at \mathbf{p}_2 . Thus C is quasismooth. Quasismoothness of $(b_6 = 0) \subset \mathbb{P}(1, 1, 3)$ can be proved in the same way. This shows (1).

The uniqueness of extremal divisorial extraction centered at a terminal quotient singular point follows from [14]. We shall consider $cAx/2$ points. By the proof of Lemma 2.7, after replacing z so that $a_6 = z^2 + f_6(x_0, x_1)$, the singularity of X' at \mathbf{p}'_1 is equivalent to

$$(y_1^2 + z^2 + g(x_0, x_1) = 0) \subset \mathbb{A}_{x_0, x_1, y_1, z}^4 / \mathbb{Z}_2(1, 1, 0, 1),$$

where the lowest degree part of g is f_6 . By (1), the polynomial f_6 cannot be a square because otherwise

$$(a_6 = 0) = (z^2 + f_6 = 0) \subset \mathbb{P}(1, 1, 3)$$

is not quasismooth. Thus by the classification [10, 13] of divisorial extraction centered at a terminal singular point of type $cAx/2$, we have the uniqueness of divisorial extraction centered at \mathbf{p}'_1 (see also [16, Section 2.2]). (2) follows for (X', \mathbf{p}'_2) by symmetry and for (X_1, \mathbf{p}_3) and (X_2, \mathbf{p}_3) since the singularities of X_1

at \mathfrak{p}_3 and of X_2 at \mathfrak{p}_3 are equivalent to those of X' at \mathfrak{p}'_2 and at \mathfrak{p}'_1 , respectively. This proves (2). \square

Lemma 2.9. *X' , X_1 and X_2 are \mathbb{Q} -factorial.*

Proof. This follows from Lemma 2.10 below. \square

Lemma 2.10. *A terminal singular point of type $cAx/2$ is (analytically) \mathbb{Q} -factorial.*

Proof. Let (X, o) be a germ of singularity of type $cAx/2$. Then X is analytically equivalent to

$$(x^2 + y^2 + g(z, t) = 0) \subset \mathbb{A}^4/\mathbb{Z}_2(0, 1, 1, 1),$$

where $g(z, t) \in (y, z)^4$ is semi-invariant. We define

$$B = \mathbb{C}[[x, y, z, t]]/(x^2 + y^2 + g(z, t))$$

and consider the \mathbb{Z}_2 action on of type $(0, 1, 1, 1)$ on B . We see that the completion $\hat{\mathcal{O}}_{X,o}$ is isomorphic to $A := B^{\mathbb{Z}_2}$. Since $o \in X$ is an isolated singularity, there is no multiple in the irreducible decomposition $g = g_1 g_2 \cdots g_d$. We see that

$$\mathrm{Cl}(B) = \bigoplus_{i=1}^d \mathbb{Z} \cdot [\mathfrak{p}_i] / \sum_{i=1}^d [\mathfrak{p}_i],$$

where $\mathfrak{p}_i = (x - \sqrt{-1}y, g_i)$ is a height 1 prime ideal of B . Let $j: \mathrm{Cl}(A) \rightarrow \mathrm{Cl}(B)$ be homomorphism induced by the injection $A \hookrightarrow B$. The image of j is contained in $\mathrm{Cl}(B)^{\mathbb{Z}_2}$ and the kernel of j is contained in $H^1(\mathbb{Z}_2, B^*)$ (cf. [9, Theorem 16.1]). The \mathbb{Z}_2 action on $\mathrm{Cl}(B)$ is given by $[\mathfrak{p}_i] \mapsto -[\mathfrak{p}_i]$. It is easy to see that $\mathrm{Cl}(B)^{\mathbb{Z}_2} = 0$ and that $H^1(\mathbb{Z}_2, B^*)$ consists of 2-torsions. It follows that $\mathrm{Cl}(A)$ consists of 2-torsions and in particular we have $\mathrm{Cl}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. This shows that (X, o) is \mathbb{Q} -factorial. \square

We consider a condition on (a_6, b_6, c_8) for X_1 and X_2 being isomorphic to each other.

Lemma 2.11. *The homomorphism $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \rightarrow H^0(X_1, \mathcal{O}_{X_1}(m))$, where $\mathbb{P} := \mathbb{P}(1, 1, 2, 3, 4, 4)$, is an isomorphism for $m \leq 5$.*

Proof. We set $X := X_1$ and let $Y = (s_0 s_1 + y b_6 + c_8 = 0) \subset \mathbb{P}$ be the weighted hypersurface containing X . Let S be the non-quasismooth locus of Y . We have $\dim S \leq 1$ since it is contained in $(s_0 = s_1 = b_6 = c_8 = 0)$. Let T be the union of S and the singular locus of \mathbb{P} . Then, $U := \mathbb{P} \setminus T$, $Y_U := Y \cap U$ and $X_U := X \cap U$ are nonsingular. Since the codimension in \mathbb{P} of each component of S is greater than or equal to 3, we have $H^i(U, \mathcal{O}_U(m)) = H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m))$ for $i = 0, 1, 2$ and for any m . This follows by considering the long exact sequence of local cohomologies. In particular, we have $H^1(U, \mathcal{O}_U(m)) = H^2(U, \mathcal{O}_U(m)) = 0$ for any m . By the long exact sequence associated to the exact sequence

$$0 \rightarrow \mathcal{O}_U(m-8) \rightarrow \mathcal{O}_U(m) \rightarrow \mathcal{O}_{Y_U}(m) \rightarrow 0,$$

we have $H^0(U, \mathcal{O}_U(m)) \cong H^0(Y_U, \mathcal{O}_{Y_U}(m))$ for $m < 8$ and $H^1(Y_U, \mathcal{O}_{Y_U}(m)) = 0$ for any m . Then, by the long exact sequence associated to the exact sequence

$$0 \rightarrow \mathcal{O}_{Y_U}(m-6) \rightarrow \mathcal{O}_{Y_U}(m) \rightarrow \mathcal{O}_{X_U}(m) \rightarrow 0,$$

we have $H^0(Y_U, \mathcal{O}_{Y_U}(m)) \cong H^0(X_U, \mathcal{O}_{X_U}(m))$ for $m < 6$. This shows that the restriction $H^0(U, \mathcal{O}_U(m)) \rightarrow H^0(X_U, \mathcal{O}_{X_U}(m))$ is an isomorphism for $m < 6$. \square

Definition 2.12. We say that a triplet (a_6, b_6, c_8) is *symmetric* if there are non-zero complex numbers α, β, γ and an automorphism τ of $\mathbb{P}(1, 1, 3)$ such that $\gamma^3 = \alpha^2 \beta^2$, $\tau^* a_6 = \alpha b_6$, $\tau^* b_6 = \beta a_6$ and $\tau^* c_8 = \gamma c_8$. A triplet (a_6, b_6, c_8) is called *asymmetric* if it is not symmetric.

Lemma 2.13. X_1 is isomorphic to X_2 if and only if (a_6, b_6, c_8) is symmetric.

Proof. Assume that there is an isomorphism $\sigma: X_1 \rightarrow X_2$. We have $\sigma^* \mathcal{O}_{X_2}(m) \cong \mathcal{O}_{X_1}(m)$ for any m since $\sigma^* K_{X_1} = K_{X_2}$. By Lemma 2.11, the sections $\sigma^* x_0, \sigma^* x_1, \sigma^* y, \sigma^* z, \sigma^* s_0, \sigma^* s_1$ can be identified with homogeneous polynomials of degree respectively 1, 1, 2, 3, 4, 4. For $i = 0, 1$, the divisor which is cut out on X_1 by $\sigma^* s_i$ passes through a singular point of type $\frac{1}{4}(1, 1, 3)$. By replacing σ with the composite of σ and the automorphism of X_1 interchanging s_0 and s_1 , we can assume that $\sigma^* s_0$ (resp. $\sigma^* s_1$) vanishes at \mathbf{p}_2 (resp. \mathbf{p}_1) and does not vanish at \mathbf{p}_1 (resp. \mathbf{p}_2). We may write $\varphi^* s_i = \alpha_i s_i + \alpha'_i y^2 + y q^{(i)} + f^{(i)}$, $\varphi^* z = \gamma z + y \ell + g$ and $\varphi^* y = \beta y + h$, where $\alpha_i, \alpha'_i, \beta, \gamma \in \mathbb{C}$, $q^{(i)}, \ell, g, h \in \mathbb{C}[x_0, x_1]$ and $f^{(i)} \in \mathbb{C}[x_0, x_1, z]$. Since the zero loci of $\varphi^*(s_0 y + s_1 y + b_6)$ and $\varphi^*(s_0 s_1 + y a_6 + c_8)$ contain X_1 , we have

$$(1) \quad \varphi^*(s_0 y + s_1 y + b_6) = \delta(s_0 y + s_1 y + a_6)$$

and

$$(2) \quad \varphi^*(s_0 s_1 + y a_6 + c_8) = \varepsilon(s_0 s_1 + y b_6 + c_8) + q(s_0 y + s_1 y + a_6)$$

for some non-zero $\delta, \varepsilon \in \mathbb{C}$ and $q \in \mathbb{C}[x_0, x_1, y]$. By comparing the terms involving s_i in (1), we have $\alpha_0 = \alpha_1$, $\beta \neq 0$ and $h = 0$. We put $\alpha := \alpha_0 = \alpha_1$. Note that there is no monomial divisible by y^3 in $\varphi^* a_6$, $\varphi^* b_6$ and $\varphi^* c_8$. By comparing terms involving s_i in (2), we have $\varepsilon = \alpha^2$, $\alpha'_i = 0$, $f^{(i)} = 0$ and $q = \alpha q^{(0)} = \alpha q^{(1)}$. By comparing terms involving y^3 in (2), we have $\ell = 0$. It follows that $\varphi^* a_6, \varphi^* b_6, \varphi^* c_8 \in \mathbb{C}[x_0, x_1, z]$. Thus, by comparing terms divisible by y^2 in (1), we have $q^{(0)} = q^{(1)} = 0$. Therefore, we have $\varphi^* s_i = \alpha s_i$ and $\varphi^* y = \beta y$, $\varphi^* z = \gamma z + g(x_0, x_1)$ and $\varphi^* x_i \in \mathbb{C}[x_0, x_1]$, and the relations $\varphi^* b_6 = \alpha \beta a_6$, $\beta \varphi^* a_6 = \alpha^2 b_6$ and $\varphi^* c_8 = \alpha^2 b_6$ are satisfied. Thus (a_6, b_6, c_8) is symmetric.

Conversely, if we are given an automorphism τ of $\mathbb{P}(1, 1, 3)$ and $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\gamma^3 = \alpha^2 \beta^2$, $\tau^* a_6 = \alpha b_6$, $\tau^* b_6 = \beta a_6$ and $\tau^* c_8 = \gamma c_8$, then the automorphism σ of $\mathbb{P}(1, 1, 2, 3, 4, 4)$ defined by $\sigma^* x_i = \tau^* x_i$ for $i = 0, 1$, $\sigma^* z = \tau^* z$ and

$$\sigma^* y = \frac{\gamma}{\beta} y, \quad \sigma^* s_0 = \frac{\alpha \beta}{\gamma} s_0, \quad \sigma^* s_1 = \frac{\alpha \beta}{\gamma} s_1,$$

restricts to an isomorphism between X_1 and X_2 . This completes the proof. \square

We show that there does exist a symmetric triplet (a_6, b_6, c_8) which satisfies Condition 2.6.

Lemma 2.14. Let a_6 and c_8 are general homogeneous polynomials in variables x_0, x_1, z . Then the triplet (a_6, a_6, c_8) is symmetric and satisfies Condition 2.6.

Proof. Let X' be the weighted hypersurface

$$X' = (y_0^2 y_1^2 + y_0 a_6 + y_1 a_6 + c_8 = 0) \subset \mathbb{P}(1, 1, 2, 2, 3)$$

and let Λ be the linear system spanned by $y_0^2 y_1^2$, M_8 and $(y_0 + y_1) M_6$, where

$$M_d := \{x_0^k x_1^l z^m \mid k, l, m \geq 0 \text{ and } k + l + 3m = d\}$$

and

$$(y_0 + y_1)M_6 := \{(y_0 + y_1)h \mid h \in M_6\}.$$

A general member X' of Λ is quasismooth outside the base locus of Λ and the base locus of Λ is the set $\{\mathfrak{p}'_1, \mathfrak{p}'_2, \mathfrak{p}'_3\}$. We see that a general X' is quasismooth at \mathfrak{p}'_3 and has a singularity of type $cAx/2$ at $\mathfrak{p}'_1, \mathfrak{p}'_2$.

Let a_6 and c_8 be general so that X' is quasismooth outside $\{\mathfrak{p}'_1, \mathfrak{p}'_2\}$ and the singularity of X' at \mathfrak{p}'_1 and \mathfrak{p}'_2 are both of type $cAx/2$. Let X be the weighted complete intersection

$$X = (s_0y + s_1y + a_6 = s_0s_1 - ya_6 - c_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4).$$

We have $X = X_1 = X_2$ and it is easy to see that X has singular points $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ which are of type $\frac{1}{4}(1, 1, 3)$, $\frac{1}{4}(1, 1, 3)$ and $cAx/2$, respectively. It remains to show that $X^\circ := X \setminus \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ is nonsingular. Instead of proving quasismoothness of X directly, we make use of the arguments in Section 3.3. There is a birational map $\sigma_{11}: X' \dashrightarrow X$ which factorizes as

$$\begin{array}{ccc} Y' & \dashrightarrow & Y \\ \varphi' \downarrow & & \downarrow \varphi \\ X' & \xrightarrow{\sigma_{11}} & X \end{array}$$

where φ' is the weighted blowup of X' at \mathfrak{p}'_1 with $\text{wt}(x_0, x_1, y_1, z) = \frac{1}{2}(1, 1, 4, 3)$, φ is the Kawamata blowup of X at \mathfrak{p}_1 and $Y' \dashrightarrow Y$ is a birational map. The construction of the above birational map is possible in the case where the singularity of X' at \mathfrak{p}'_1 is of type $cAx/2$ and that of X at \mathfrak{p}_1 is of type $\frac{1}{4}(1, 1, 3)$. Let $\Delta' \subset Y'$ and $\Delta \subset Y$ be proper transforms of $(y_1 = a_6 = c_8 = 0) \subset X'$ and $(y = s_1 = a_6 = c_8 = 0) \subset X$, respectively. Then the birational map $Y' \dashrightarrow Y$ induces an isomorphism $Y' \setminus \Delta' \cong Y \setminus \Delta$. Y' has three singular points whose types are $\frac{1}{3}(1, 1, 2)$, $\frac{1}{4}(1, 1, 3)$ and $cAx/2$, where $\frac{1}{4}(1, 1, 3)$ is the point contained in the exceptional divisor of φ' , and each singular point is not contained in Δ' . On the other hand, Y has at least three singular points of the same types, where the singular point of type $\frac{1}{3}(1, 1, 2)$ is contained in the exceptional divisor of φ , and each point is not contained in Δ . Since $Y' \setminus \Delta' \cong Y \setminus \Delta$, we see that X° is nonsingular outside $(y = s_1 = a_6 = c_8 = 0)$. By considering automorphism of X interchanging s_0 and s_1 , we see that X° is nonsingular outside $(y = s_0 = a_6 = c_8 = 0)$.

It is then enough to show that X is nonsingular along $S := (y = s_0 = s_1 = a_6 = c_8 = 0)$. We see that the restriction to S of the Jacobian matrix of the affine C_X of X can be written as

$$J_{C_X}|_S = \begin{pmatrix} \frac{\partial a_6}{\partial x_0} & \frac{\partial a_6}{\partial x_1} & 0 & \frac{\partial a_6}{\partial z} & 0 & 0 \\ -\frac{\partial c_8}{\partial x_0} & -\frac{\partial c_8}{\partial x_1} & 0 & -\frac{\partial c_8}{\partial z} & 0 & 0 \end{pmatrix}.$$

Therefore X is quasismooth along S since the complete intersection $(a_6 = c_8 = 0)$ in $\mathbb{P}(1, 1, 3)$ is quasismooth for general a_6 and c_8 by Theorem 2.4. Thus X is nonsingular along S and this completes the proof. \square

2.3. Maximal singularities and the structure of proof. We give the definition of maximal singularities and Sarkisov links. For a linear system \mathcal{H} and a divisor D on a normal projective \mathbb{Q} -factorial variety, we say that \mathcal{H} is \mathbb{Q} -linearly

equivalent to D , denoted by $\mathcal{H} \sim_{\mathbb{Q}} D$, if a member of \mathcal{H} is \mathbb{Q} -linearly equivalent to D .

Definition 2.15. Let $\mathcal{H} \sim_{\mathbb{Q}} -nK_X$ be a movable linear system on a \mathbb{Q} -Fano variety with Picard number one. A *maximal singularity* of \mathcal{H} is an extremal divisorial extraction $\varphi: Y \rightarrow X$ having exceptional divisor E with

$$\frac{1}{n} > c(X, \mathcal{H}) = \frac{a_E(K_X)}{\text{mult}_E(\mathcal{H})},$$

where $\text{mult}_E(\mathcal{H})$ is the multiplicity of \mathcal{H} along E , $a_E(K_X)$ is the discrepancy of K_X along E and

$$c(X, \mathcal{H}) := \max\{\lambda \mid K_X + \lambda\mathcal{H} \text{ is canonical}\}$$

is the canonical threshold of \mathcal{H} . We say that $\varphi: Y \rightarrow X$ is a *maximal singularity* (without referring a linear system) if it is a maximal singularity of some movable linear system \mathcal{H} . A subvariety $\Gamma \subset X$ is called a *maximal center* if there is a maximal singularity $\varphi: Y \rightarrow X$ whose center is Γ .

Definition 2.16. A *Sarkisov link* between \mathbb{Q} -Fano varieties X and X' with Picard number one is a birational map $\sigma: X \dashrightarrow X'$ which factorizes as

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ \varphi \downarrow & & \downarrow \varphi' \\ X & \xrightarrow{\sigma} & X' \end{array}$$

where φ, φ' are extremal divisorial extractions and $Y \dashrightarrow Y'$ is a composite of inverse flips, flops and flips (in that order). We sometimes call σ the *Sarkisov link starting with* the extremal divisorial extraction φ . The center on X of φ is called the *center* of the Sarkisov link σ .

The remainder of this paper is devoted to the proof the following.

Theorem 2.17. Let (a_6, b_6, c_8) be a triplet of homogeneous polynomials in x_0, x_1, z satisfying Condition 2.6 and X', X_1, X_2 the \mathbb{Q} -Fano threefolds corresponding to (a_6, b_6, c_8) . Then no nonsingular point and no curve on X', X_1 and X_2 is a maximal center. Moreover, the existence of Sarkisov links from X', X_1 and X_2 centered at singular points are described as follows.

- (1) There exist Sarkisov links $X' \dashrightarrow X_1$ and $X' \dashrightarrow X_2$ centered at the $cAx/2$ points \mathbf{p}'_1 and \mathbf{p}'_2 , respectively.
- (2) There exists a Sarkisov link $X' \dashrightarrow X'$ centered at the $\frac{1}{3}(1, 1, 2)$ point \mathbf{p}'_3 of X' which is a birational involution.
- (3) For $i = 1, 2$, there exists a Sarkisov link $X_i \dashrightarrow X'$ centered at each $\frac{1}{4}(1, 1, 3)$ point of X_i .
- (4) For the $cAx/2$ points $\mathbf{p}_3 \in X_1$ and $\mathbf{p}_3 \in X_2$, one of the following holds.
 - (a) Neither $\mathbf{p}_3 \in X_1$ nor $\mathbf{p}_3 \in X_2$ is a maximal center.
 - (b) There exists a Sarkisov link $X_1 \dashrightarrow X_2$ centered at $\mathbf{p}_3 \in X_1$ and its inverse $X_2 \dashrightarrow X_1$ is centered at $\mathbf{p}_3 \in X_2$.

In view of the fact that there is a unique divisorial extraction centered at each singular point of X', X_1 and X_2 , Theorem 1.3 follows from Lemma 2.13 and Theorem 2.17 by [16, Lemma 2.32]. The construction of Sarkisov links will be

given in Section 3 and exclusion of nonsingular points and curves will be done in Sections 4 and 5.

3. SARKISOV LINKS

We construct various Sarkisov links between X' , X_1 and X_2 . Throughout this section we assume that (a_6, b_6, c_8) satisfies Condition 2.6.

3.1. Birational involution of X' . We shall construct a birational involution ι of X' which is a Sarkisov link centered at the $\frac{1}{3}(1, 1, 2)$ point \mathbf{p}'_3 . After re-scaling y_0, y_1, z , we may assume that the coefficients of z^2 in a_6 and b_6 are both 1. We write $a_6 = z^2 + zf_3 + f_6$, $b_6 = z^2 + zg_3 + g_6$ and $c_8 = z^2h_2 + zh_5 + h_8$, where $f_i, g_i, h_i \in \mathbb{C}[x_0, x_1]$. It follows that the defining polynomial of X' is

$$(y_0 + y_1 + h_2)z^2 + (y_0f_3 + y_1g_3 + h_5)z + y_0^2y_1^2 + y_0f_6 + y_1g_6 + h_8.$$

Let Z' be the weighted hypersurface in $\mathbb{P}(1, 1, 2, 2, 5)$ with homogeneous coordinates x_0, x_1, y_0, y_1, t , where $\deg t = 5$, defined by the equation

$$t^2 + (y_0f_3 + y_1g_3 + h_5)t + (y_0 + y_1 + h_2)(y_0^2y_1^2 + y_0f_6 + y_1g_6 + h_8) = 0.$$

This equation is obtained by multiplying $y_0 + y_1 + h_2$ to the defining equation of X' and then identifying t with $(y_0 + y_1 + h_2)z$. We have a birational map $X' \dashrightarrow Z'$ by identifying t with $(y_0 + y_1 + h_2)z$. Let $\varphi': Y' \rightarrow X'$ be the Kawamata blowup of X' at \mathbf{p}'_3 . Then φ' resolves the indeterminacy of $X' \dashrightarrow Z'$ and the induced birational morphism $\psi': Y' \rightarrow Z'$ is a flopping contraction which contracts finitely many curves. Let $\iota_{Z'}: Z' \rightarrow Z'$ be the biregular involution interchanging the fibers of the double cover $Z' \rightarrow \mathbb{P}(1, 1, 2, 2)$. Then $\psi'^{-1} \circ \iota_{Z'} \circ \psi': Y' \dashrightarrow Y'$ is the flop and we have the following Sarkisov link.

Proposition 3.1. *The diagram*

$$\begin{array}{ccccc} Y' & \overset{\iota_{Y'}}{\dashrightarrow} & Y' & & \\ \varphi' \downarrow & \searrow \psi' & & \swarrow \psi' & \downarrow \varphi' \\ X' & & Z' & \xrightarrow{\iota_{Z'}} & Z' & & X' \end{array}$$

is a Sarkisov link centered at \mathbf{p}'_3 and the induced map $\iota: X' \dashrightarrow X'$ is a birational involution.

3.2. Link between X_1 and X_2 . For $i = 1, 2$, let $\varphi_i: Y_i \rightarrow X_i$ be the weighted blowup of X_i at the $cAx/2$ point \mathbf{p}_3 with $\text{wt}(x_0, x_1, z, s_0, s_1) = \frac{1}{2}(1, 1, 3, 4, 4)$ and $\pi_i: X_i \dashrightarrow \mathbb{P}(1, 1, 3, 4, 4)$ the projection to the coordinates x_0, x_1, z, s_0 and s_1 . The images of π_1 and π_2 are the same and it is the weighted hypersurface

$$Z := ((s_0 + s_1)(s_0s_1 - c_8) + a_6b_6 = 0) \subset \mathbb{P}(1, 1, 3, 4, 4).$$

The sections x_0, x_1, z, s_0 and s_1 on X_i lift to plurianticanonical sections on Y_i and they define the morphism $\psi_i: Y_i \rightarrow Z$ such that $\psi_i = \varphi_i \circ \pi_i$. It follows that ψ_i is a K_{Y_i} -trivial contraction. We see that ψ_i contracts the proper transform on Y_i of

$$\Delta := (s_0 + s_1 = s_0s_1 - c_8 = a_6 = b_6 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4, 4).$$

We see that $\dim \Delta = 2$ if and only if a_6 is proportional to b_6 , that is, there is a non-zero $\lambda \in \mathbb{C}$ such that $a_6 = \lambda b_6$.

Lemma 3.2. *If a_6 is proportional to b_6 , then the $cAx/2$ point of X_i is not a maximal center for $i = 1, 2$.*

Proof. Since φ_i is the unique extremal divisorial extraction centered at the $cAx/2$ point of X_i , it is enough to show that φ_i is not a maximal center. Let E_i be the exceptional divisor of φ_i . We have $K_{Y_i} = \varphi_i^* K_{X_i} + (1/2)E_i$. Note that Δ is a surface since a_6 is proportional to b_6 . It follows that ψ_i contracts a divisor. Let C be an irreducible and reduced curve on Y_i contracted by ψ_i . Then, $(-K_{Y_i} \cdot C) = 0$ and

$$(E_i \cdot C) = 2(K_{Y_i} \cdot C) - 2(\varphi_i^* K_{X_i} \cdot C) = -2(\varphi_i^* K_{X_i} \cdot C) > 0$$

since C is not contracted by φ_i . This shows that there are infinitely many curves on Y_i which intersect $-K_{Y_i}$ non-positively and E_i positively. It follows from [16, Lemma 2.19] that \mathfrak{p} is not a maximal center. \square

Proposition 3.3. *Assume that a_6 is not proportional to b_6 . Then the diagram*

$$\begin{array}{ccccc} Y_1 & \dashrightarrow & & & Y_2 \\ \varphi_1 \downarrow & \searrow \psi_1 & & \swarrow \psi_2 & \downarrow \varphi_2 \\ X_1 & & Z & & X_2 \end{array}$$

gives a Sarkisov link $\theta: X_1 \dashrightarrow X_2$ centered at the $cAx/2$ point of X_1 . The inverse $\theta^{-1}: X_2 \dashrightarrow X_1$ is a Sarkisov link centered at the $cAx/2$ point of X_2 .

Proof. By the assumption, $\dim \Delta = 1$ and thus ψ_i is a flopping contraction since ψ_i is a K_{Y_i} -trivial contraction whose exceptional locus is the proper transform of $\Delta \subset X_i$. The birational map $\theta = \pi_2^{-1} \circ \pi_1: X_1 \dashrightarrow X_2$ is given by

$$(x_0:x_1:y:z:s_0:s_1) \mapsto (x_0:x_1:\frac{b_6}{a_6}y:z:s_0:s_1)$$

and it cannot be biregular unless b_6/a_6 is a constant. We see that b_6/a_6 cannot be a constant because otherwise Δ is of 2-dimensional. It follows that $\psi_2^{-1} \circ \psi_1: Y_1 \dashrightarrow Y_2$ is a flop and thus $\theta: X_1 \dashrightarrow X_2$ is a Sarkisov link. \square

Remark 3.4. Note that a_6 being proportional to b_6 implies that (a_6, b_6, c_8) is symmetric. It follows that X_1 and X_2 are connected by a Sarkisov link whenever X_1 is not isomorphic to X_2 .

3.3. Links between X' and X_i . We construct Sarkisov links between X' and X_i for $i = 1, 2$. Recall that

$$\mathfrak{p}'_1 = (0:0:1:0:0) \text{ and } \mathfrak{p}'_2 = (0:0:0:1:0)$$

are the $cAx/2$ points of X' and

$$\mathfrak{p}_1 = (0:0:0:0:1:0) \text{ and } \mathfrak{p}_2 = (0:0:0:0:0:1)$$

are the $\frac{1}{4}(1, 1, 3)$ points of X_i . Let $\mathbb{P} := \mathbb{P}(1, 1, 2, 3, 4)$ be the weighted projective space with homogeneous coordinates x_0, x_1, y, z, s and let $\pi'_1: X' \dashrightarrow \mathbb{P}$ be the rational map defined by

$$(x_0:x_1:y_0:y_1:z) \mapsto (x_0:x_1:y_1:z:y_0y_1).$$

By multiplying y_1 to the defining polynomial of X' and then replacing y_1 with y and y_0y_1 with s , we see that the image of π'_1 is the weighted hypersurface

$$Z_1 = (s^2y + sa_6 + y^2b_6 + yc_8 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4),$$

and $\pi'_1: X' \dashrightarrow Z_1$ is a birational map defined outside \mathbf{p}'_1 .

Let $\pi_1: X_1 \dashrightarrow \mathbb{P}$ be the projection defined by

$$(x_0:x_1:y:z:s_0:s_1) \mapsto (x_0:x_1:y:z:s_1)$$

which is defined outside \mathbf{p}_1 . By considering the ratio

$$s_0 = -\frac{s_1 y + a_6}{y} = \frac{y b_6 + c_8}{s_1},$$

we see that the image of π_1 is Z_1 and $\pi_1: X_1 \dashrightarrow Z_1$ is birational. We define $\sigma_{11} := \pi_1^{-1} \circ \pi'_1: X' \dashrightarrow X_1$.

Let $\eta_1: X_1 \rightarrow X_1$ be the automorphism of X_1 which interchanges s_0 and s_1 and we define $\sigma_{12} := \eta_1 \circ \sigma_{11}: X' \dashrightarrow X_1$. By the symmetry between y_0 and y_1 , the same construction gives a birational map $\sigma_{21}: X' \dashrightarrow X_2$ and $\sigma_{22} := \eta_2 \circ \sigma_{21}: X' \dashrightarrow X_2$, where η_2 is the automorphism of X_2 which interchanges s_0 and s_1 .

Proposition 3.5. *For $i = 1, 2$ and $j = 1, 2$, the birational map $\sigma_{ij}: X' \dashrightarrow X_i$ is a Sarkisov link centered at the $cAx/2$ point \mathbf{p}'_i and the inverse $\sigma_{ij}^{-1}: X_i \dashrightarrow X'$ is a Sarkisov link centered at the $\frac{1}{4}(1, 1, 3)$ point \mathbf{p}_j .*

Proof. We shall prove the assertion for σ_{11} . The rest follows by symmetry.

Let $\varphi'_1: Y'_1 \rightarrow X'$ be the weighted blowup of X' at \mathbf{p}'_1 with $\text{wt}(x_0, x_1, y_1, z) = \frac{1}{2}(1, 1, 4, 3)$. Note that φ'_1 is the unique extremal divisorial extraction of centered at \mathbf{p}'_1 . We see that x_0, x_1, y_1, z and $y_0 y_1$ lift to plurianticanonical sections on Y' and φ'_1 resolves the indeterminacy of π'_1 . Thus we have a $K_{Y'}$ -trivial birational morphism $\psi'_1: Y' \rightarrow Z$. Let $\varphi_1: Y_1 \rightarrow X_1$ the Kawamata blowup of X_1 at \mathbf{p}_1 . We see that x_0, x_1, y, z, s_1 lift to plurianticanonical sections on Y_1 and φ_1 resolves the indeterminacy of π_1 . Thus we have a K_{Y_1} -trivial birational morphism $\psi_1: Y_1 \rightarrow Z$ and we have the commutative diagram

$$\begin{array}{ccccc} Y'_1 & \dashrightarrow & & & Y_1 \\ \varphi'_1 \downarrow & \searrow \psi'_1 & & \swarrow \psi_1 & \downarrow \varphi_1 \\ X' & & Z & & X_1 \end{array}$$

It remains to show that $Y'_1 \dashrightarrow Y_1$ is a flop, that is, both ψ'_1 and ψ_1 are small.

We see that ψ'_1 contracts the proper transform of $(y_1 = a_6 = c_8 = 0) \subset X'$ to $S := (y = a_6 = c_8 = s = 0) \subset Z$, and ψ_1 contracts the proper transform of $(y = s_1 = a_6 = c_8 = 0) \subset X_1$ to S . Therefore ψ'_1 is divisorial if and only if ψ_1 is so, and this is equivalent to the assertion that a_6 and c_8 share a common component. Assume that a_6 and c_8 have a component $d \in \mathbb{C}[x_0, x_1, z]$. Then, since $(a_6 = 0) \subset \mathbb{P}(1, 1, 3)$ is quasismooth, we have $d = a_6$. Hence $c_8 = a_6 e_2$ for some $e_2 \in \mathbb{C}[x_0, x_1, z]$. Let $C = (y = s_0 = s_1 = a_6 = 0)$ be a curve. We see that $C \subset X_1$ and the restriction of the Jacobian matrix of the affine cone of X_1 to C is of the form

$$J_{C_{X_1}}|_C = \begin{pmatrix} \frac{\partial a_6}{\partial x_0} & \frac{\partial a_6}{\partial x_1} & 0 & \frac{\partial a_6}{\partial z} & 0 & 0 \\ -\frac{\partial a_6}{\partial x_0} e_2 & -\frac{\partial a_6}{\partial x_1} e_2 & -b_6 & -\frac{\partial a_6}{\partial z} e_2 & 0 & 0 \end{pmatrix}.$$

This shows that X_1 is not quasismooth along $C \cap (b_6 = 0)$. This is a contradiction and thus $Y'_1 \dashrightarrow Y_1$ is a flop. \square

4. EXCLUDING MAXIMAL CENTERS ON X'

In this section let (a_6, b_6, c_8) be a triplet satisfying Condition 2.6. We shall exclude all the nonsingular points and curves on X' as maximal singularity.

4.1. Nonsingular points. In this subsection we exclude nonsingular points of X' as maximal center.

Definition 4.1. Let X be a normal projective variety embedded in a weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ with homogeneous coordinates x_0, \dots, x_n and $\mathbf{p} \in X$ a nonsingular point. We say that a set $\{g_i\}$ of homogeneous polynomials in x_0, \dots, x_n *isolates* \mathbf{p} if \mathbf{p} is an isolated component of

$$X \cap \bigcap_i (g_i = 0).$$

We say that a Weil divisor L *isolates* \mathbf{p} if there is an integer $s > 0$ such that \mathbf{p} is an isolated component of the base locus of the linear system

$$\mathcal{L}_{\mathbf{p}}^s := |\mathcal{I}_{\mathbf{p}}^s(sL)|.$$

Lemma 4.2 ([7]). *Let X be a \mathbb{Q} -Fano 3-fold with Picard number one and $\mathbf{p} \in X$ a nonsingular point. If $-lK_X$ isolates \mathbf{p} for some $l \leq 4(-K_X)^3$, then \mathbf{p} is not a maximal center.*

Proof. We refer the reader to [7, Proof of (A)] and also to [16, Lemma 2.14] for a proof. \square

The following enables us to find a divisor which isolates a nonsingular point.

Lemma 4.3 ([7, Lemma 5.6.4]). *Let X be a normal projective variety embedded in $\mathbb{P}(a_0, \dots, a_n)$ and $\{g_i\}$ a set of homogeneous polynomials of $\deg g_i = l_i$. If a set $\{g_i\}$ of polynomials isolates \mathbf{p} , then lA isolates \mathbf{p} , where $l = \max\{l_i\}$.*

Proposition 4.4. *No nonsingular point on X' is a maximal center.*

Proof. Let $\mathbf{p} = (\xi_0 : \xi_1 : \eta_0 : \eta_1 : \zeta)$ be a nonsingular point of X' . If $\xi_0 \neq 0$, then the set

$$\{\xi_0 x_1 - \xi_0 x_0, \xi_0^2 y_0 - \eta_0 x_0^2, \xi_0^2 y_1 - \eta_1 x_0^2, \xi_0^3 w - \zeta x_0^3\}$$

isolates \mathbf{p} and thus $3A$ isolates \mathbf{p} . Similarly, $3A$ isolates \mathbf{p} if $\xi_1 \neq 0$. Assume that $\xi_0 = \xi_1 = 0$. In this case, at least one of η_0 and η_1 is non-zero since \mathbf{p} is not a singular point. Without loss of generality, we may assume that $\eta_0 \neq 0$. Then the set

$$\{x_0, x_1, \eta_0 y_1 - \eta_0 y_0, \eta_0^3 z^2 - \zeta^2 y_0^3\}$$

isolates \mathbf{p} and thus $6A$ isolates \mathbf{p} . Therefore Lemma 4.2 shows that \mathbf{p} is not a maximal center since $3 < 6 \leq 4/(A^3) = 6$. \square

4.2. Curves. In this subsection we exclude curves on X' as maximal center.

Lemma 4.5. *No curve on X is a maximal center except possibly for a curve of degree $1/2$ which does not pass through the $\frac{1}{3}(1, 1, 2)$ point \mathbf{p}_3 .*

Proof. Let $\Gamma \subset X'$ be a curve. By [16, Lemma 2.9], Γ can be a maximal center only if $(A \cdot \Gamma) < (A^3) = 2/3$. If Γ passes through the $\frac{1}{3}(1, 1, 2)$ point \mathbf{p}'_3 , then it is not a maximal singularity since there is no extremal divisorial extraction centered along a curve passing through a terminal quotient singular point. If

Γ does not pass through \mathbf{p}'_3 , then $(A \cdot \Gamma) \in \frac{1}{2}\mathbb{Z}$. This follows since the divisor $(y_0 + y_1 = 0)_{X'} \sim_{\mathbb{Q}} 2A$ intersects Γ at nonsingular points of X' and thus $(2A \cdot \Gamma) \in \mathbb{Z}$. Combining the above arguments, Γ is not a maximal center unless it satisfies $(A \cdot \Gamma) = 1/2$ and $\mathbf{p}'_3 \notin \Gamma$. \square

Let Γ be a curve of degree $1/2$ on X' which does not pass through \mathbf{p}'_3 . Since Γ passes through a $cAx/2$ point, we may assume that $\mathbf{p}'_1 \in \Gamma$ without loss of generality. The defining polynomial of X' is $F' := y_0^2 y_1^2 + y_0 a_6 + y_1 b_6 + c_8$. After re-scaling y_0, y_1, z , we may assume that the coefficients of z^2 in a_6 and b_6 are both 1.

Lemma 4.6. *We have $\Gamma = (x_1 = y_1 = z = 0)$ after replacing x_0, x_1, z .*

Proof. The restriction $\pi|_{\Gamma}: \Gamma \rightarrow \pi(\Gamma)$ of the projection $\pi: X' \dashrightarrow \mathbb{P}(1, 1, 2, 2)$ from \mathbf{p}'_3 is a finite morphism. We have $1/2 = \deg(\pi|_{\Gamma}) \deg(\pi(\Gamma))$ and $\deg \pi(\Gamma) \in \frac{1}{4}\mathbb{Z}$. We claim that $\deg \pi(\Gamma) = 1/2$ and $\pi|_{\Gamma}$ is an isomorphism. If $\deg \pi(\Gamma) = 1/4$, then $\pi(\Gamma) = (x_0 = x_1 = 0)$. It follows that

$$\Gamma \subset (x_0 = x_1 = 0)_{X'} = (x_0 = x_1 = y_0^2 y_1^2 + y_0 z^2 + y_1 z^2).$$

We see that $(x_0 = x_1 = 0)_{X'}$ is an irreducible and reduced curve of degree $2/3$. This is a contradiction and the claim is proved.

After replacing x_0, x_1 , we may assume that $\pi(\Gamma) = (x_1 = \theta_0 y_0 + \theta_1 y_1 - \lambda x_0^2 = 0)$ for some $\theta_0, \theta_1, \lambda \in \mathbb{C}$. Since $\mathbf{p}'_1 \in \Gamma$, $\pi(\Gamma)$ passes through $(0:0:1:0) \in \mathbb{P}(1, 1, 2, 2)$. This implies that $\theta_0 = 0$ and then we may assume that $\theta_1 = 1$. Since $\deg \Gamma = 1/2$ and $\Gamma \subset (x_0 = y_1 - \lambda x_0^2 = 0)_{X'}$, we have $\Gamma = (x_1 = y_1 - \lambda x_0^2 = z - \mu y_0 x_0 - \nu x_0^3 = 0)$ for some $\mu, \nu \in \mathbb{C}$. Replacing $z \mapsto z + \nu x_0^3$, we assume that $\nu = 0$. Now it is straightforward to see that Γ is indeed contained in X' if and only if $\lambda = \mu = 0$, $x_0^6 \notin a_6$ and $x_0^8 \notin c_8$. This completes the proof. \square

We write $a_6 = z^2 + z f_3(x_0, x_1) + f_6(x_0, x_1)$. We have $f_6(x_0, 0) = c_8(x_0, 0, 0) = 0$ since $\Gamma = (x_1 = y_1 = z = 0)$ is contained in X' . We write $f_6 = x_1 f_5$.

Lemma 4.7. *At least one of f_3 and f_5 is not divisible by x_1 .*

Proof. Assume that both f_3 and f_5 are divisible by x_1 . Let $F_1 := s_0 y + s_1 y + a_6$ be the defining polynomial of X_1 of degree 6. If both f_3 and f_5 are divisible by x_1 , then $\partial F_1 / \partial x_0, \partial F_1 / \partial x_1, \partial F_1 / \partial y, \partial F_1 / \partial z, \partial F_1 / \partial s_0$ and $\partial F_1 / \partial s_1$ vanish at the point $(1:0:0:0:0:0) \in X_1$. This cannot happen since X_1 is quasismooth outside its $cAx/2$ point. \square

Let $\mathcal{M} \subset |3A|$ be the linear system spanned by the sections $x_0^2 x_1, x_0 x_1^2, x_1^3, y_1 x_0, y_1 x_1, z$, and let S be a general member of \mathcal{M} . We have $\text{Bs } \mathcal{M} = \Gamma \cup \{\mathbf{p}'_2\}$, $\text{Bs } \mathcal{M}_{y_1} = (x_0 = x_1 = 0)_{X'} \not\supset \Gamma$, $\text{Bs } \mathcal{M}_{x_1} = (x_0 = x_1 = y_1 = 0)_{X'}$. By Lemma 2.5, S is nonsingular along $\Gamma \setminus \{\mathbf{p}'_1\}$.

Lemma 4.8. *We have $(\Gamma^2) \leq -3/2$.*

Proof. The section which cuts out S on X can be written as $z + x_1 q + \alpha_0 y_1 x_0 + \alpha_1 y_1 x_1$, where $q = q(x_0, x_1)$ is a quadric and $\alpha_0, \alpha_1 \in \mathbb{C}$. We work on the open subset on which $w \neq 0$. Let $\varphi: T \rightarrow S$ be the weighted blowup of S at \mathbf{p}_2 with $\text{wt}(x_0, x_1, y_1, z) = \frac{1}{2}(1, 1, 4, 3)$, E its exceptional divisor and $\tilde{\Gamma}$ the proper transform of Γ on T . We claim that $E = E_1 + E'$, where E_1 is a prime divisor,

E' does not contain E_1 as a component, $(\tilde{\Gamma} \cdot E_1) = 1$ and $\tilde{\Gamma}$ is disjoint from the support of E' . Indeed we have the isomorphisms

$$\begin{aligned} E &\cong (z^2 + zf_3 + x_1f_5 = z + x_1q = 0) \subset \mathbb{P}(1, 1, 4, 3) \\ &\cong (x_1^2q^2 - x_1qf_3 + x_1f_5 = 0) \subset \mathbb{P}(1, 1, 4). \end{aligned}$$

We set $E_1 = (x_1 = 0)$ and $E' = (x_1q^2 - qf_3 + f_5 = 0)$. Since at least one of f_3 and f_5 is not divisible by x_1 and q is general, we see that E' does not contain E_1 as a component and E' is disjoint from $\tilde{\Gamma}$. Moreover, E_1 intersects $\tilde{\Gamma}$ transversally at a nonsingular point. This proves the claim.

We write $\varphi^*\Gamma = \tilde{\Gamma} + rE_1 + F$ for some rational number r and an effective \mathbb{Q} -divisor F whose support is contained in $\text{Supp } E'$. We have $r \leq 1/2$ since the section x_1 cuts out on S the curve Γ and another curve, and x_1 vanishes along E_1 to order $1/2$. An explicit computation shows that $K_T = \varphi^*K_S - E$. We have

$$(\Gamma^2) = (\varphi^*\Gamma \cdot \tilde{\Gamma}) = (\tilde{\Gamma}^2) + (rE_1 + F \cdot \tilde{\Gamma}) = (\tilde{\Gamma}^2) + r$$

and

$$(\tilde{\Gamma}^2) = -(K_T \cdot \tilde{\Gamma}) - 2 = -(K_S \cdot \Gamma) - 1.$$

Combining these with $(K_S \cdot \Gamma) = 2 \deg \Gamma = 1$, we get $(\Gamma^2) = -2 + r \leq -3/2$. \square

Proposition 4.9. *No curve on X' is a maximal center.*

Proof. By Lemma 4.5, it is enough to exclude a curve Γ degree $1/2$ which does not pass through \mathfrak{p}'_3 . We keep preceding notation. We assume that Γ is a maximal center. Then there is a movable linear system $\mathcal{H} \subset |nA|$ on X such that $\text{mult}_\Gamma \mathcal{H} > n$. Let S be a general member of \mathcal{M} so that we have

$$A|_S \sim \frac{1}{n} \mathcal{H}|_S = \frac{1}{n} \mathcal{L} + \gamma \Gamma,$$

where \mathcal{L} is the movable part of $\mathcal{H}|_S$ and $\gamma \geq \text{mult}_\Gamma \mathcal{H}/n > 1$. This is possible since the base locus of \mathcal{M} does not contain a curve other than Γ . Let L be a \mathbb{Q} -divisor on S such that $nL \in \mathcal{L}$. Note that $(L^2) \geq 0$ since L is nef. We get

$$(L^2) = (A|_S - \gamma \Gamma)^2 = 3(A^3) - 2(\deg \Gamma)\gamma + (\Gamma^2)\gamma^2 = 2 - \gamma + (\Gamma^2)\gamma^2.$$

Since $(\Gamma^2) < -3/2$ by Lemma 4.8 and $\gamma > 1$, we have

$$(L^2) < 2 - 1 + (\Gamma^2) \leq -1/2.$$

This is a contradiction and Γ is not a maximal center. \square

5. EXCLUDING MAXIMAL CENTERS ON X_1 AND X_2

In this section let (a_6, b_6, c_8) be a triplet satisfying Condition 2.6. We exclude nonsingular points and curves on X , where X is either X_1 or X_2 .

5.1. Nonsingular points. The following excludes all the nonsingular points on X .

Proposition 5.1. *No nonsingular point on X is a maximal center.*

Proof. Let $\mathbf{p} = (\xi_0 : \xi_1 : \eta : \zeta : \sigma_0 : \sigma_1)$ be a nonsingular point of X . If $\xi_0 \neq 0$, then the set

$$\{\xi_1 x_0 - \xi_0 x_1, \xi_0^2 y - \eta x_0^2, \xi_0^3 z - \zeta x_0^3, \xi_0^4 s_0 - \sigma_0 x_0^4, \xi_0^4 s_1 - \sigma_1 x_0^4\}$$

isolates \mathbf{p} and thus $4A$ isolates \mathbf{p} . Similarly, if $\xi_1 \neq 0$, then $4A$ isolates \mathbf{p} . Assume that $\xi_0 = \xi_1 = 0$. If further $\eta = 0$, then \mathbf{p} is a singular point of type $\frac{1}{4}(1, 1, 3)$. Hence $\eta \neq 0$ and the set

$$\Lambda := \{x_0, x_1, \eta^2 s_0 - \sigma_0 y^2, \eta^2 s_1 - \sigma_1 y^2\}$$

isolates \mathbf{p} . Here, to see that Λ indeed isolates \mathbf{p} , we consider the projection $\pi: X \rightarrow \mathbb{P}(1, 1, 2, 4, 4)$ to the coordinates x_0, x_1, y, s_0, s_1 , which does not contract a curve. The intersection of zero loci of polynomials in Λ is the fiber $\pi^{-1}(\pi(\mathbf{p}))$ which is a finite set of points including \mathbf{p} . This shows that Λ isolates \mathbf{p} . It follows that $4A$ isolates \mathbf{p} . By Lemma 4.2, \mathbf{p} is not a maximal center since $4 < 4/(A^3) = 8$. \square

5.2. Curves.

Proposition 5.2. *No curve on X is a maximal center.*

Proof. Let Γ be an irreducible curve on X . If Γ passes through a singular point of type $\frac{1}{4}(1, 1, 3)$, then there is no extremal divisorial contraction centered along Γ , hence Γ cannot be a maximal center. If Γ does not pass through a $\frac{1}{4}(1, 1, 3)$ point, then $(2A \cdot \Gamma)$ is a positive integer and thus $(A \cdot \Gamma) \geq 1/2$. By [16, Lemma 2.9], \mathbf{p} is not a maximal center since $(A^3) = 1/2$. This completes the proof. \square

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